

# Renormalization-Group Theory and Universality along the Lambda Line of $^4\text{He}$

V. Dohm

Institut für Theoretische Physik, Technische Hochschule Aachen  
D-52056 Aachen, Germany

## Abstract

The present status of the renormalization-group (RG) predictions on the superfluid density  $\rho_s$  and the specific heat  $C^+$  and  $C^-$  near the  $\lambda$ -line of  $^4\text{He}$  is briefly reviewed. Particular attention is given to universal amplitude ratios related to these quantities. The goals of a new theory project are presented that involves higher-order calculations of the amplitude functions of  $\rho_s$ ,  $C^+$  and  $C^-$  which are of fundamental importance for a test of the universality predictions of the RG theory.

## 1 Introduction

One of the fundamental achievements of the renormalization-group (RG) theory of critical phenomena is the identification of universality classes in terms of the dimensionality  $d$  of the system and the number  $n$  of components of the order parameter [1]. Specifically, RG theory predicts that, within a given universality class, the critical exponents, certain amplitude ratios and scaling functions are universal quantities that do not depend, e.g., on the strength of the interaction or on thermodynamic variables (such as the pressure). The superfluid transition of  $^4\text{He}$  belongs to the  $d = 3, n = 2$  universality class and provides a unique opportunity for an experimental test of the universality prediction by means of measurements of the critical behavior at various pressures  $P$  along the  $\lambda$ -line  $T_\lambda(P)$  (and along the  $\lambda$ -line of

$^3\text{He} - ^4\text{He}$  mixtures). Early tests have been performed by Ahlers and collaborators and consistency with the universality prediction was found within the experimental resolution [2]. At a significantly higher level of accuracy, the superfluid density  $\rho_s$  and the specific heat  $C^+$  and  $C^-$  (or, equivalently, thermal expansion coefficient  $\beta^\pm$ ) above and below  $T_\lambda(P)$  are planned to be measured in the Superfluid Universality Experiment (SUE) [3] under micro-gravity conditions or at reduced gravity in the low-gravity simulator [4]. As demonstrated recently [5], this would allow to perform measurements up to  $|t| \simeq 10^{-9}$  in the reduced temperature  $t = (T - T_\lambda(P))/T_\lambda(P)$ .

Important steps towards an improved universality test can already be performed, even in the presence of gravity, by new ground-based measurements provided that non-universal and universal effects are properly separated in a nonlinear RG analysis of the data [6]. To extract the leading critical exponents of  $\rho_s$  and  $C^\pm$  from the experimental data and to demonstrate their universality at a highly quantitative level requires detailed knowledge on certain universal amplitude ratios and on non-asymptotic corrections to the asymptotic power laws. These quantities can be calculated by means of field-theoretic RG methods applied to the  $\varphi^4$  model [1, 6].

In the following we point to a serious lack of quantitative knowledge in the theoretical literature [1] on these amplitude ratios and correction terms. We also discuss the implications of very recent two-loop results for  $\rho_s$  [7] and shall summarize the goals of a new theory project [8] on higher-order RG calculations of the amplitude functions of  $\rho_s$  and  $C^\pm$ .

## 2 Universal amplitude ratios

In the previous analyses of experimental data for  $\rho_s$  and  $C^\pm$  (or  $\beta^\pm$ ) the following representations were employed [2, 5, 9]

$$\rho_s = k_B T_\lambda (m/\hbar)^2 A_\rho (1 + k_1 |t|) |t|^\zeta (1 + a_\rho |t|^\Delta) \quad , \quad (1)$$

$$C^+ = \frac{A^+}{\alpha} |t|^{-\alpha} (1 + a_c^+ |t|^\Delta + E^+ |t|) + B, \quad t > 0 \quad (2)$$

$$C^- = \frac{A^-}{\alpha} |t|^{-\alpha} (1 + a_c^- |t|^\Delta + E^- |t|) + B, \quad t < 0 \quad (3)$$

where  $m$  is the helium mass. These representations contain 10 nonuniversal amplitudes  $A_\rho, k_1, a_\rho, A^\pm, a_c^\pm, E^\pm, B$  which need to be adjusted at each pressure. Their values vary by 15 – 30%, typically, along the  $\lambda$ -line. The exponents  $\alpha$  and  $\zeta = \nu(d - 2) = (2 - \alpha)(d - 2)/d$  and the Wegner exponent  $\Delta$  are predicted to be universal. The existing theoretical predictions on these exponents [10] are based on field-theoretic calculations to five-loop order [11] and Borel resummation. The recent experimental result [5]  $\alpha = -0.01285 \pm 0.00038$  is consistent with but more accurate than the RG estimate [10]. In addition, RG theory predicts the amplitude ratios

$$A^+/A^-, \quad (A^-)^{1/d}/A_\rho, \quad a_c^-/a_\rho, \quad a_c^+/a_c^- \quad (4)$$

to be universal [1]. Knowing their universal values for the  $d = 3$ ,  $n = 2$  universality class would impose significant constraints on the analysis of the experimental data and would thereby improve the reliability and precision of the experimental results regarding the universality of  $\alpha$  and  $\zeta$ .

The present theoretical knowledge of the ratios (4), however, is based only on low-order (mainly 1- and 2-loop) field-theoretic calculations [1] which imply an uncertainty at the level of at least 10 – 30%. For example, even the sign of the ratio  $a_c^-/a_\rho$  was incorrectly predicted by the two-loop  $\varepsilon (= 4 - d)$  expansion [12]. This issue was resolved by a (low-order) calculation within the  $d = 3$  field theory [13]. The present status of the experimental and theoretical values of the universal ratios (4) is summarized in Table 1 (without error estimates).

**Table 1** Universal amplitude ratios for the  $n = 2, d = 3$  universality class

	$A^+/A^-$	$(A^-)^{1/3}/A_\rho$	$a_c^-/a_\rho$	$a_c^+/a_c^-$
Experiment [2]	1.067	0.85	-0.068	1.03
Experiment [14]	1.088			0.85
$d = 3$ Field Theory [13]	1.05	0.78	-0.045	1.6
$\varepsilon$ -Expansion [1]	1.029	1.0	1/6	1.17

As noted in Ref. 13, there is a large uncertainty of the theoretical value for  $a_c^+/a_c^-$ . The  $d = 3$  field-theoretical values for  $(A^-)^{1/3}/A_\rho$  and  $a_c^-/a_\rho$  are based on one-loop results [13]. Very recent  $d = 3$  two-loop results [7], to be discussed in the subsequent section, imply considerably different values [15], thus indicating a considerable lack of reliability of low-order results for the amplitude ratios (4).

### 3 Recent results on $\rho_s$ in two-loop order

In an effort to improve the theoretical prediction on these amplitude ratios, a two-loop calculation of  $\rho_s$  was recently carried out by Burnett, Strösser and Dohm within the  $\varphi^4$  field theory in  $d = 3$  dimensions [7]. The superfluid density can be calculated via Josephson's definition [16]

$$\rho_s = k_B T (m/\hbar)^2 |\psi_0|^2 \frac{\partial}{\partial k^2} \chi_T(k)^{-1} /_{k=0} \quad (5)$$

where  $\chi_T(k)$  is the transverse susceptibility at finite wave number  $k$  and  $\psi_0$  is the order parameter (Bose condensate wave function). Within the  $d = 3$   $\varphi^4$  field theory the representation (5) becomes [7, 13]

$$\rho_s = k_B T (m/\hbar)^2 \xi_-^{-1} f_\psi(u) f_T(u) \quad (6)$$

where

$$\xi_- = \xi_0^- |t|^{-\nu} (1 + a_\xi^- |t|^\Delta + \dots) \quad (7)$$

is an appropriately defined correlation length. The amplitude functions  $f_\psi(u)$  and  $f_T(u)$  depend on the effective renormalized  $\varphi^4$  coupling  $u = u(\xi_-^{-1})$  which for  $\xi_- \rightarrow \infty$  approaches the fixed point value  $u(0) = u^* = 0.0362$  [17]. These functions are plotted in Fig. 1 in zero-, one-, and two-loop order. Their fixed point values  $f_\psi(u^*)$  and  $f_T(u^*)$  determine the asymptotic amplitude  $A_\rho$  in (1) while their derivatives at  $u = u^*$  contribute to the subleading amplitude  $a_\rho$ .

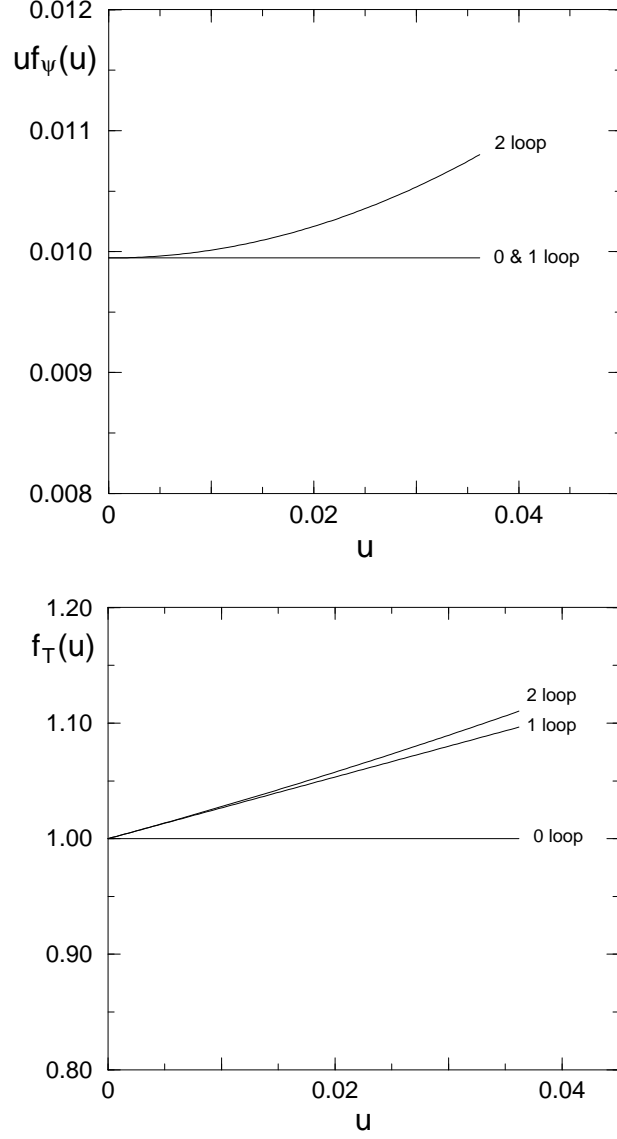


Fig. 1: Amplitude functions  $uf_\psi(u)$  and  $f_T(u)$  for the order parameter and for the  $k^2$  part of the inverse of the transverse susceptibility in zero-, one- and two-loop order vs the renormalized coupling  $u$ . The curves terminate at the fixed point  $u^* = 0.0362$ . From Ref. 7.

While the two-loop correction to  $f_T$  is remarkably small (about 1%) at the fixed point  $u^* = 0.0362$ , the two-loop contribution to  $f_\psi(u)$  is about 10 %, thus indicating a considerable uncertainty of low-order perturbation theory. The uncertainty is even larger for the derivative of  $f_\psi(u)$  at  $u^*$  which implies a correspondingly large uncertainty of  $a_\rho$  in (1) and of the ensuing amplitude ratio  $a_c^-/a_\rho$ . A similar uncertainty exists with regard to the amplitude function  $F_-(u)$  of the specific heat  $C^-$  below  $T_\lambda$  [7, 13] determining  $A^-$  and  $a_c^-$  in (3) as well as with regard to the amplitude function  $G(u)$  of the helicity modulus  $\Upsilon$  [7] which is proportional to  $\rho_s$ .

## 4 New theory project

In order to significantly reduce the uncertainty of the theoretical predictions on the amplitude ratios (4) it has been proposed [8] to perform new higher-order field-theoretic RG calculations and Borel resummations of various amplitude functions. In general, the results of such calculations depend on the specific renormalization scheme employed. In view of various applications, the method of the minimally renormalized  $\varphi^4$  field theory at fixed dimension  $d = 3$  [7, 13, 17] appears to be most appropriate. The following quantities need to be calculated:

- (i) The additive renormalization constant  $A(u, \varepsilon)$  of the specific heat and the related RG function  $B(u)$ . At present,  $A(u, \varepsilon)$  and  $B(u)$  are known only in two-loop order [17]. For a three-loop calculation within a different renormalization scheme see [20].
- (ii) The amplitude function  $f_\psi(u)$  of the square of the order parameter appearing in Eq. (6). For  $n = 1$  it is known up to five-loop order and in Borel-resummed form [18] but for  $n > 1$  only up to two-loop order [7].
- (iii) The amplitude function  $F_-(u)$  of the specific heat below  $T_\lambda$ . For  $n = 1$  it is known up to five-loop order and in Borel-resummed form [18] but for  $n > 1$  only the two-loop result is known at present [13]. (Even for  $n = 1$  the five-loop result needs to be revised on the basis of higher-order calculations for  $B(u)$  according to item (i) above.)

- (iv) The amplitude function  $F_+(u)$  of the specific heat above  $T_\lambda$ . Previously this function has been computed in Borel-resummed form for  $n = 1, 2, 3$  on the basis of six-loop results [19]. But this calculation needs to be revised on the basis of higher-order calculations for  $B(u)$  according to item (i) above.
- (v) The amplitude function  $f_\xi^-(u)$  of an appropriately defined correlation length  $\xi_-$  below  $T_\lambda$ . This correlation length performs the task of absorbing logarithms in the four-point coupling  $u_0$  in the bare perturbation series in  $d = 3$  dimensions [7, 13, 18]. For  $n = 1$ ,  $\xi_-$  can be taken to be the ordinary correlation length of three-dimensional Ising-like systems below  $T_c$ . For a three-loop calculation of  $\xi_-$  for  $n = 1$  within a different renormalization scheme see [21].
- (vi) The amplitude function  $f_T(u)$  [see Eq. (6)] of the  $k^2$  part of the inverse of the transverse correlation function  $\chi_T(k)$ . At present this function is known in two-loop order [7]. Equivalently (and preferably), the amplitude function  $G(u) = 4\pi f_\psi(u)f_T(u)$  of the helicity modulus  $\Upsilon$  [7] should be calculated in higher order.

Higher-order calculations of  $G(u)$  or  $f_T(u)$  would be more demanding than those of  $f_\psi$  and  $F_-$  because they involve perturbation theory at finite wave number  $k$ . In view of the smallness of the two-loop correction to  $f_T(u)$  (see Fig. 1) it has been argued [7] that this low-order result for  $f_T(u)$  can be considered as rather reliable, presumably within a few percent. In a first step of future calculations it will therefore be sufficient to confine the project to the quantities (i) - (v) noted above.

Once the functions  $B(u)$ ,  $f_\psi(u)$ ,  $F_-(u)$ ,  $F_+(u)$  and  $f_\xi^-(u)$  are known at an accuracy of a few percent it will be possible to determine the amplitude ratios (4) at the same level of accuracy. It will be advantageous to go beyond the asymptotic representations (1) - (3) and, instead, include the entire Wegner series, i.e. higher-order terms  $|t|^{n\Delta}$ ,  $n = 2, 3, \dots$ , *without introducing new nonuniversal parameters into the analysis*. For the strategy of such a *nonlinear RG analysis* see Refs. 6 and 17. This may imply that the phenomenological analytic corrections  $\sim k_1|t|$  and  $\sim E^\pm|t|$  in the representations (1) - (3) are unnecessary or will turn out to be much smaller than determined



previously [2, 5, 9].

The success of this theory project depends on the order of perturbation theory up to which the calculations can be carried out. Preliminary considerations by Larin [22] indicate that the amplitude functions  $f_\psi(u)$ ,  $F_-(u)$  and  $f_\xi^-(u)$  can be determined up to four-loop order, and the RG functions  $A(u, \varepsilon)$  and  $B(u)$ , on the basis of previous work [11], up to five-loop order. Such results will have an important impact on a future test of the fundamental RG prediction of critical-point universality.

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### **Note added**

Very recently a five-loop calculation of  $A(u, \varepsilon)$  and of  $B(u)$  has been performed for general  $n$  and a Borel resummation of  $B(u)$  has been carried out for  $n = 1, 2, 3$  [23, 24]. The Borel resummed function  $B(u)$  differs from the two-loop result  $B(u) = \frac{n}{2} + O(u^2)$  by less than 1 % at the fixed point  $u^*$ . For  $n = 2$  the result is  $B(u^*) = 1.0053 \pm 0.0022$ . Furthermore, a three-loop calculation of the amplitude functions  $f_\psi(u)$  and  $F_-(u)$  in three dimensions below  $T_c$  has been carried out for general  $n \geq 1$  [25, 26].

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